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A Fuzzy Set Approach for Generalized CRR Model: An Empirical Analysis of S&P 500 Index Options*

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Abstract. This paper applies fuzzy set theory to the Cox, Ross and Rubinstein (CRR) model to set up the fuzzy binomial *option pricing model* (OPM). The model can provide reasonable ranges of option prices, which many investors can use it for arbitrage or hedge. Because of the CRR model can provide only theoretical reference values for a generalized CRR model in this article we use fuzzy volatility and fuzzy riskless interest rate to replace the corresponding crisp values. In the fuzzy binomial OPM, investors can correct their portfolio strategy according to the right and left value of triangular fuzzy number and they can interpret the optimal difference, according to their individual risk preferences. Finally, in this study an empirical analysis of S&P 500 index options is used to find that the fuzzy binomial OPM is much closer to the reality than the generalized CRR model.

Key words: fuzzy set theory, fuzzy binomial OPM, option pricing model (OPM), a generalized CRR model, triangular fuzzy number, portfolio strategy

JEL Classification:

1. Introduction

As derivative-based financial products become a major part of current global financial market, it is imperative to bring the basic concepts of options, especially the pricing method to a level of standardization in order to eliminate possible human negligence in the content or structure of the option market. Recently, the optimal option price has been used to compute by the binomial model or the Black-Scholes model. However, volatility and riskless

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interest rate are assumed as constant in those models. Hence, many subsequent studies emphasized the estimated riskless interest rate and volatility. Cox (1975) introduced the concept of Constant-Elasticity-of-Variance for volatility. Hull and White (1987) released the assumption that the distribution of price of underlying asset and volatility are constant. Wiggins (1987), Scott (1987), Lee, Lee and Wei (1991) released the assumption that the volatility is constant and assumed that the volatility followed Stochastic-Volatility. Amin (1993) and Scott (1997) considered that the Jump-Diffusion process of stock price and the volatility were random process.

Researchers have so far made substantial effort and achieve significant results concerning the pricing of options (e.g., Brennan and Schwartz, 1977; Geske and Johnson, 1984; Barone-Adesi and Whaley, 1987). Empirical studies have shown that given their basic assumptions, existing pricing model seem to have difficulty in properly handling the uncertainties inherent in any investment process. This may have to do with the fact that traditional option pricing models have failed to include fuzzy factors in the analysis. So, most of these studies have focused on how to release the assumptions in the CRR model and the B-S model, including: (1) the short-term riskless interest rate is constant, (2) the volatility of a stock is constant. After loosening these assumptions, we need to set up new model.

In reality, the future state of a system might not be known completely due to lack of information, so investment problems are often uncertain or vague in a number of ways. This type of uncertainty has long been handled appropriately by probability theory or statistics. However, in many areas, such as funds, stocks, debt, derivates and others, human judgment of events may be significantly different based on individuals' subjective perceptions or personality tendencies for judgment, evaluation and decisions, thus it is often fuzzy. So we will use fuzzy set theory to describe and eliminate the "Fuzziness" which is the subjective assessment made by investors in the OPM.

Therefore, this paper attempts to apply fuzzy set theory to the generalized CRR model, in order to replace the complex models of previous studies. If we can predict the optimal range of an option price, investors can make a profit and hedge from that option. When the uncertainties in the investment environment are taken into consideration, the riskless interest rate and volatility attributes facing the investors are not only incompliant with the basic assumptions of the B-S model, but there are cases whereby the model does not even apply. Hence, this study provides an analysis of investment practice in order to examine the feasibility of using the fuzzy binomial OPM in practice.

The remainder of this paper is organized as follows. The generalized CRR OPM is introduced and discussed in Section 2. In Section 3 the fuzzy binomial OPM is inferred when $t = 1, 2, \ldots, n$. of option price. In Section 4 an empirical analysis is illustrated to assess the accuracy of approximation to the CRR model. A comparison generalized with fuzzy binomial OPM in Section 5, and conclusions are presented in Section 6.

2. Generalized CRR models: A review

In this section, we briefly review the binomial OPM of Cox, Ross and Rubinstein (CRR) model in 1979 under fixed parameters. The n-period binomial OPM may be written as

$$C = (1/r^n) \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \, \mathrm{Max}[0, u^k d^{n-k} S - X] \tag{1}$$

where

$C =$ the n-period option price;
$X =$ the option striking price;
$S =$ the current stock price;
$n =$ the number of periods to maturity;
$d =$ one plus the percentage of downward movement in stock price;
$u =$ one plus the percentage of upward movement in stock price;
$r =$ one plus the riskless interest rate per period;
$p = (r - d)/(u - d)$.

It is assumed that $u > r > d$; thus, $0 < p < 1$. Let m be an integer such that

$$u^{m-1} d^{n-(m-1)} S \leq E < u^m d^{n-m} S.$$

Cox, Ross and Rubinstein (CRR) used a binomial model to derive a formula (1) for OPM. The binomial OPM is a discrete-time model, whereas the B-S model is continuous-time model. When the duration of an option is divided into infinite time slots, the CRR model is closer to the B-S model.

2.1. *The assumptions of the CRR model*

The CRR model is based upon two fundamental assumptions for the price calculation of options: firstly, that the riskless interest rate is given; and secondly, that the stock-market volatility is constant for the duration of the options. Hence, the investor has to ensure that the variables in the formula, namely the attributes of the riskless interest rate and the stock-market volatility are in compliance with the assumptions of the model for real-life applications. Failure to do so will result in price inaccuracy. So we suggest an ideal OPM based on integrating fuzzy set theory with the CRR model (see Appendix A).

2.2. *The comments of riskless interest rate*

Although the risk neutrality assumption serves as a basis for the construction of the OPM, it provides only "instant" validity. Beyond the "instant" time frame, the assumption either no longer exists or any attempt to extend the validity of the assumption becomes fundamentally impractical. More importantly, in the realm of financial economics, the concept of "risk neutrality" should only be confined to the discussion of the demand side of a stock market, rather than associating it with the stock-market "equilibrium prices," as the B-S model did. The B-S model began its formulation by composing for the present moment a risk-free portfolio of stocks and their corresponding options in an appropriate proportion. The problem is that the state of zero risk for such an investment portfolio exists only in a single instant. According to Hull (1993), for options with a relatively long expiration, transcendental deductions are invalid for drawing conclusions on the actual content of the entire options. When we discount stock prices at option expiration with a riskless interest rate, we are in fact declaring that all those different types of stocks will grow at the same riskless interest rate. This will inevitably lead to biased results.

The best way to modify the assumption of OPM that all investors are "risk neutral" individuals—treating the research and analysis environment of the model as a "risk neutral" world—in pricing stock options is to simply define this assumption as an "objective" view, so that it can be applied to each individual investor. In the real world, it is often given that each investor possesses some subjectivity in making decisions and describes objectivity issues with subjectively interpreted actions, the meaning of "risk preference" can vary from one person to another. In other words, the concept lacks complete "objectivity" or "standardization", so it cannot be used in any way to measure the preferences for a particular stock of individual investors.[1] Industry forecasts its future riskless interest rate in accordance with the classification of "booming economy", "fair economy", or "depression". The definitions of "booming economy", "fair economy", and "depression", depend on the investor's subjective opinion. This means that past discussions on "risk preference" theories that were based on some kind of complete "objectivity" assumption are not validated. Therefore, this study introduces the concept of triangular fuzzy number to explain riskless interest rate including booming economy: R_h (the highest riskless interest rate), depression: R_l (the lowest riskless interest rate), and fair economy: R_m (the medium riskless interest rate), which is between R_l and R_h riskless interest rate.

2.3. *The type of volatility*

Theoretically, the volatility in any OPM should be the future volatility. If future volatility is known, we will know the distribution of future option price and the true optimal price of option. Hence, many researches use many ways to estimate future volatility. Frequently we use the volatility of stock return to substitute future volatility. Some of the more popular methods are described as follows:

(1) Historical Volatility: Assume that past volatility is same as future volatility; we can then use the volatility of a stock return to estimate its future volatility. This is the most general way, also called Equally-Weighted Moving Average method.

$$\sigma = \sqrt{\frac{1}{N}\sum_{t=1}^{n}(r_t - \bar{r})^2}$$

r_t: stock return of t-th period; $\bar{r}$: the average of stock return of $t = 1$ to n period.

(a) For r_t is discrete time $r_t = \frac{S_t - S_{t-1}}{S_{t-1}}$, ex-dividend not considered, and when return is not stationary, then σ is bigger;
(b) For r_t is Continuous time $r_t = \ln(\frac{S_t}{S_{t-1}})$, ex-dividend considered, and when return is stationary, then σ is smaller; where, S_t: the closing price of t-th day; S_{t-1}: the closing price of $t-1$-th day.

(2) Implied Volatility (Implied Standard Deviation): If we assure that the OPM is correct, we can use an inverse function to compute implied volatility, and we will get different volatility over the same period of time. The volatility of deep-in-the-money ($S \gg X$) and deep-out-of-the-money ($S \ll X$) is greater than that of at-the-money ($S = X$), which is called the volatility smile. Investors can buy an option with lower implied volatility and a sell option with higher implied volatility.
(3) Parkinson method: Parkinson (1980) proposed that the use of only closing prices could not represent the actual situations, so he used the highest and lowest prices instead of the closing price.
(4) Garman and Klass method: Garman and Klass (1980) modified the Parkinson method, adding the opening and closing price to it. They used four variables to estimate the volatility: closing price, opening price, highest price, and lowest price.
(5) ARCH method: Bollerslev (1986) introduced the concept of time series to the volatility of stock price, considering the volatility of stock price that would change over time instead of constant. And the stock price has the characteristic of clutching. He formulated an OPM with the current volatility (σ_t^2) affected by the volatility of last period (σ_{t-1}^2) and the stock return of last period (r_{t-1}^2). The Nobel Prize shared by Engle and Granger (2003) for methods of analyzing economic time series with time-varying volatility (ARCH) and common trends (cointegration). Engle found that the concept of autoregressive conditional heteroskedasticity (ARCH) accurately captures the properties of many time series and developed methods for statistical modeling of time-varying volatility.

3. Fuzzy binomial OPM

Options normally include call option and put option. The owner of a call (put) has the right to buy (sell) something, at a set price, within a set time period in exchange for this right. There are five primary factors affecting option prices. These are striking price, current stock price, time, riskless interest rate, and volatility. Since the striking price and time until option expiration are both determined, current stock prices reflect on ever period, but riskless interest rate determined the interest rate of currency market, and volatility can't be observed directly but can be estimated by historical data and situation analysis. Therefore, riskless interest rate and volatility are estimated. We can use the concept of fuzziness to estimate the two factors riskless interest rate and volatility. The fuzzy binomial OPM is illustrated below, including one-step and inference of fuzzy binomial OPM.

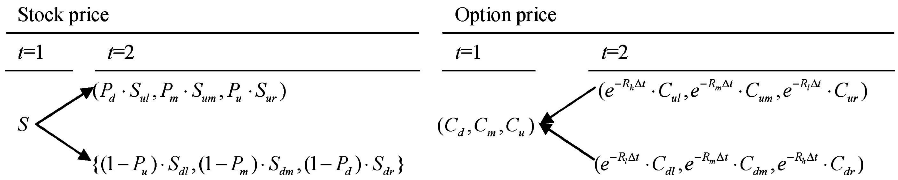

Figure 1. Stock price and call price movement in time (Δt) under the fuzzy binomial OPM.

3.1. *One step fuzzy binomial OPM*

According to the triangular fuzzy number, we can set up the stock price, call price, volatility, riskless interest rate, and probability that have three values including low, medium, and high. The deriving process of call prices is illustrated as Figure 1.

Assume that an option will mature in a given period; and the stock price is initially set to $S(t = 1)$. It can move up or down in the next period ($t = 2$). It is possible that there are three scenarios in the up movement, $S_{ur}(=ur \cdot S)$, $S_{um}(=um \cdot S)$, or $S_{ul}(=ul \cdot S)$, and three scenarios in the down movement, $S_{dr}(=dr \cdot S)$, $S_{dm}(=dm \cdot S)$, or $S_{dl}(=dl \cdot S)$. Hence, in each time interval the stock price moves from its initial value of S to one of the above six new values, S_{ur}, S_{um}, S_{ul}, S_{dr}, S_{dm} or S_{dl}.
where,

S_{ur}: stock price of up movement under greatest volatility, $S_{ur} = S_0 \cdot [e^{(1+\rho)\sigma\sqrt{\Delta t}}]$,
S_{um}: stock price of up movement under medium volatility, $S_{um} = S_0 \cdot [e^{\sigma\sqrt{\Delta t}}]$,
S_{ul}: stock price of up movement under smallest volatility, $S_{ul} = S_0 \cdot [e^{(1-\rho)\sigma\sqrt{\Delta t}}]$,
S_{dr}: stock price of down movement under smallest volatility, $S_{dr} = S_0 \cdot [e^{-(1-\rho)\sigma\sqrt{\Delta t}}]$,
S_{dm}: stock price of down movement under medium volatility, $S_{dm} = S_0 \cdot [e^{-\sigma\sqrt{\Delta t}}]$,
S_{dl}: stock price of down movement under greatest volatility, $S_{dl} = S_0 \cdot [e^{-(1+\rho)\sigma\sqrt{\Delta t}}]$.

The call prices in the maturity are $C_{ur} = \max(S_{ur} - X, 0)$, $C_{um} = \max(S_{um} - X, 0)$, $C_{ul} = \max(S_{ul} - X, 0)$, $C_{dr} = \max(S_{dr} - X, 0)$, $C_{dm} = \max(S_{dm} - X, 0)$, $C_{dl} = \max(S_{dl} - X, 0)$.

We can combine the bond and stock to get the call price. We make an example of the greatest volatility.

$$C_u = \Delta_u \cdot S + B_u$$

where,

Δ_u: the number of stock purchased under the greatest volatility, suppose stocks can be divided infinitely;
S: current stock price;
B_u: amount invested in bonds under the greatest volatility.

Assume we buy Δ_u number of stocks under the greatest volatility, and invest amount B_u in risk assets, (R_l, R_m, R_h), after one period, the value of portfolio under for up movement is $\Delta_u \cdot ur \cdot S + e^{R_h\Delta t} \cdot B_u$ and the one for down is $\Delta_u \cdot dl \cdot S + e^{R_h\Delta t} \cdot B_u$.

where,

ur: the range of the greatest volatility of "up" movement. The movement from S to $S_{ur}(=ur \cdot S)$ is therefore an "up" movement under the greatest volatility;
dl: the range of the greatest volatility of "down" movement. The movement from S to $S_{dl}(=dl \cdot S)$ is therefore a "down" movement under the greatest volatility;
$e^{R_l \Delta t}$: the discount factor of riskless interest rate;
$e^{R_l \Delta t} \cdot B_u$: the sum of riskless interest rate and principal of bond after one period under the greatest volatility.

Suppose the value of the portfolio either under up or down movement is equal to the value of the corresponding call price. Given $\hat{ur} \in ur$, $\hat{dl} \in dl$, then

$$\Delta_u \cdot \hat{ur} \cdot S + e^{R_h \Delta t} \cdot B_u = C_{ur} \tag{2}$$
$$\Delta_u \cdot \hat{dl} \cdot S + e^{R_h \Delta t} \cdot B_u = C_{dl} \tag{3}$$

From equations (2) and (3), we can get

$$B_u = \frac{\hat{ur} \cdot C_{dl} - \hat{dl} \cdot C_{ur}}{(\hat{ur} - \hat{dl})e^{R_h \Delta t}} \tag{4}$$
$$\Delta_u = \frac{C_{ur} - C_{dl}}{(\hat{ur} - \hat{dl})S}, \quad \text{hedge ratio under greatest volatility.} \tag{5}$$

We replace Δ_u and B_u in $C_u = \Delta_u \cdot S + B_u$, by equations (4) and (5), and we can derive the following equations.

$$\begin{aligned}
C_u &= \frac{C_{ur} - C_{dl}}{(\hat{ur} - \hat{dl})S} \cdot S + \frac{\hat{ur} \cdot C_{dl} - \hat{dl} \cdot C_{ur}}{\hat{ur} \cdot e^{R_h \Delta t} - \hat{dl} \cdot e^{R_h \Delta t}} \\
&= \frac{C_{ur} \cdot e^{R_h \Delta t} - C_{dl} \cdot e^{R_h \Delta t}}{\hat{ur} \cdot e^{R_h \Delta t} - \hat{dl} \cdot e^{R_h \Delta t}} + \frac{\hat{ur} \cdot C_{dl} - \hat{dl} \cdot C_{ur}}{\hat{ur} \cdot e^{R_h \Delta t} - \hat{dl} \cdot e^{R_h \Delta t}} \\
&= \frac{(e^{R_h \Delta t} - \hat{dl})C_{ur} + (\hat{ur} - e^{R_h \Delta t})C_{dl}}{\hat{ur} \cdot e^{R_h \Delta t} - \hat{dl} \cdot e^{R_h \Delta t}} \\
&= \frac{(e^{R_h \Delta t} - \hat{dl})C_{ur} + (\hat{ur} - e^{R_h \Delta t})C_{dl}}{(\hat{ur} - \hat{dl})e^{R_h \Delta t}}
\end{aligned}$$

Let

$$P_u = \frac{e^{R_h \Delta t} - \overset{\wedge}{dl}}{\overset{\wedge}{ur} - \overset{\wedge}{dl}}, \quad \text{then } 1 - P_u = \frac{\overset{\wedge}{ur} - \overset{\wedge}{dl} - e^{R_h \Delta t} + \overset{\wedge}{dl}}{\overset{\wedge}{ur} - \overset{\wedge}{dl}} = \frac{\overset{\wedge}{ur} - e^{R_h \Delta t}}{\overset{\wedge}{ur} - \overset{\wedge}{dl}}.$$

Similarly

$$\text{Let } P_m = \frac{e^{R_m \Delta t} - \overset{\wedge}{dm}}{\overset{\wedge}{um} - \overset{\wedge}{dm}}, \quad \text{then } 1 - P_m = \frac{\overset{\wedge}{um} - e^{R_m \Delta t}}{\overset{\wedge}{um} - \overset{\wedge}{dm}}.$$

$$\text{Let } P_d = \frac{e^{R_l \Delta t} - \overset{\wedge}{dr}}{\overset{\wedge}{ul} - \overset{\wedge}{dr}}, \quad \text{then } 1 - P_d = \frac{\overset{\wedge}{ul} - e^{R_l \Delta t}}{\overset{\wedge}{ul} - \overset{\wedge}{dr}}.$$

The probabilities of three up movements are P_u, P_m and P_d, while the probabilities of three down movement is $1 - P_u$, $1 - P_m$ and $1 - P_d$, respectively. In the above equations, C_u, C_m and C_d are the right, medium and the left values of the current fuzzy option price. So we can get the fuzzy binomial OPM under greatest volatility. By the above formula, in a one-step fuzzy binomial OPM, let (C_u, C_m, C_d) be the triangular fuzzy number for the current call price, with C_u, C_m and C_d be the greatest, medium and smallest volatility respectively. Then

$$\begin{aligned} C_u &= e^{-R_l \Delta t}[P_u \cdot C_{ur} + (1 - P_u) \cdot C_{dl}]; \\ C_m &= e^{-R_m \Delta t}[P_m \cdot C_{ur} + (1 - P_m) \cdot C_{dl}]; \\ C_d &= e^{-R_h \Delta t}[P_d \cdot C_{ul} + (1 - P_d) \cdot C_{dr}]. \end{aligned}$$

3.2. *Two-step fuzzy binomial OPM*

Step 1. The two-step fuzzy OPM can be further processed, as follows. The fuzzy of stock price is combined at $t = 3$, the different paths of stock price may generate the same values. The numbers of every node at different periods will be inferred later.

Step 2. Next, we use the stock price at $t = 3$ and exercise price to compute the option price and we move from the stock price of $t = 3$ to the call price of $t = 3$. We obtain the call price from the stock price, and then we infer the call price from $t = 3$ to $t = 2$. Similar to the process $t = 3$ to $t = 2$, we obtain the call price at $t = 1$. Finally, the method for defuzzified a fuzzy ranking (Opricovic and Tzeng, 2003), and the BNP (Best Nonfuzzy Performance) value for the triangular fuzzy number $\tilde{R}_i(DR_t, MR_t, UR_t)$ can be found using the following equation:

$$BNP_t = DR_t + [(UR_t - DR_t) + (MR_t - DR_t)]/3 \quad \forall t \tag{5}$$

After using equation (5), the call price is still the only triangular fuzzy number. We can follow the same process to get a triangular fuzzy number for current call prices at $t = 2$, and we can compute the interval value of the call price at $t = 1$.

3.3. N-step fuzzy binomial OPM

In order to obtain the option price from the stock price, the detail and inference process are derived and explained as below.

Step 1. Let ρ be the percentage of upswing and downing relative to the volatility σ, for reference, we call ρ the sensitivity index of volatility. Estimate ρ and σ value to compute the S_{ur}, S_{um}, S_{ul}, S_{dr}, S_{dm} or S_{dl}. We estimated ρ and σ to compute volatility. In the other words, $\sigma(1+\rho)$ is the volatility of the right value of up movement, $\sigma(1-\rho)$ is the volatility of the left value of up movement, and $S_{ur}(=ur\cdot S)$, $S_{um}(=um\cdot S)$, and $S_{ul}(=ul\cdot S)$ are the degree of up movement. In the down movement, $-\sigma(1-\rho)$ is the volatility of the right value of down movement, $-\sigma\ (1+\rho)$ is the volatility of the left value of down movement, and $S_{dr}(=dr\cdot S)$, $S_{dm}(=dm\cdot S)$, and $S_{dl}(=dl\cdot S)$ are the degree of down movement.

Step 2. From the stock price of $t=n$ to the call price of $t=n$. We use the stock price at $t=n$ and exercise price to compute the option price. The process is the same with CRR model. $C_n = \max(S_n - X, 0)$, where, C_n is the call price at time t, S_n is the stock price at time t, X is striking price.

Step 3. The call price from $t=n$ to $t=n-1$. Now we integrate the fuzzy set with the CRR model, and use different probabilistic for different volatilities and risk interest rates. Here, C_u, C_m and C_d are the largest, middle, and smallest values of the fuzzy option price. Each node of the call price has three call prices, not just a single one, as in the CRR model. So we replace the call prices C_{ur} and C_{dl} for the greatest volatility to get $C_{ut} = e^{-R_l\Delta t}[P_u\cdot C_{ur} + (1-P_u)\cdot C_{dl}]$; C_{um} and C_{dm} for the medium volatility to get $C_{mt} = e^{-r_m\Delta t}[C_{um}\cdot P_m + C_{dm}\cdot(1-P_m)]$; and C_{ul} and C_{dr} for the smallest volatility to get $C_{dt} = e^{-R_h\Delta t}[C_{ul}\cdot P_d + C_{dr}\cdot(1-P_d)]$.

Step 4. The call price from $t=n-1$ to $t=n-2$. We use the same principle with the call price from $t=n$ to $t=n-1$, however with three call prices at $t=n-1$. We compute the BNP value of triangular fuzzy number to operate, and a triangular fuzzy number represents the call price at every node. If we still use the same way for the call price from $t=n$ to $t=n-1$, the value we get at every node will be multiple call price.

Step 5. The call price from $t=n-2$ to $t=n-3,\ldots,t-1$. The same principle applies to the call price from $t=n-1$ to $t=n-2$, for computing the BNP value to get the reasonable call price from equation (5), we can further formulate equation (5) into equation (6). Hence, we can get the only triangular fuzzy number of the current call price, not multiple solutions.

$$C_t = C_{dt}[(C_{ut}-C_{dt}) + (C_{mt}-C_{dt})]/3 \tag{6}$$

As mentioned earlier, we know that the movement of stock in the fuzzy binomial OPM is the same as in the CRR model, it either goes up or down. But in the fuzzy binomial OPM, there are three scenarios for each “up” and “down”. These are S_{ur}, S_{um}, S_{ul}, S_{dr}, S_{dm} or S_{dl}. The value of every node in the fuzzy tree of stock price is the initial stock S with ur, *um*, *ul*, *dr*, *dm*, or *dl*, the difference of different periods and nodes is the order of them. We call them that a, b, c, d, e, f: $[e^{(1+\rho)\sigma\sqrt{\Delta t}}]^a$, $[e^{\sigma\sqrt{\Delta t}}]^b$, $[e^{(1-\rho)\sigma\sqrt{\Delta t}}]^c$, $[e^{-(1-\rho)\sigma\sqrt{\Delta t}}]^d$, $[e^{-\sigma\sqrt{\Delta t}}]^e$, $[e^{-(1+\rho)\sigma\sqrt{\Delta t}}]^f$.

We can conclude that the value of every node at different periods is as follows:

1. $S_0 \cdot [e^{(1+\rho)\sigma\sqrt{\Delta t}}]^a\,[e^{\sigma\sqrt{\Delta t}}]^b$	2. $S_0 \cdot [e^{(1+\rho)\sigma\sqrt{\Delta t}}]^a\,[e^{(1-\rho)\sigma\sqrt{\Delta t}}]^c$	3. $S_0 \cdot [e^{(1+\rho)\sigma\sqrt{\Delta t}}]^a\,[e^{-(1-\rho)\sigma\sqrt{\Delta t}}]^d$
4. $S_0 \cdot [e^{(1+\rho)\sigma\sqrt{\Delta t}}]^a[e^{-\sigma\sqrt{\Delta t}}]^e$	5. $S_0 \cdot [e^{(1+\rho)\sigma\sqrt{\Delta t}}]^a[e^{-(1+\rho)\sigma\sqrt{\Delta t}}]^f$	6. $S_0 \cdot [e^{\sigma\sqrt{\Delta t}}]^b[e^{(1-\rho)\sigma\sqrt{\Delta t}}]^c$
7. $S_0 \cdot [e^{\sigma\sqrt{\Delta t}}]^b[e^{-(1-\rho)\sigma\sqrt{\Delta t}}]^d$	8. $S_0 \cdot [e^{\sigma\sqrt{\Delta t}}]^b[e^{-\sigma\sqrt{\Delta t}}]^e$	9. $S_0 \cdot [e^{\sigma\sqrt{\Delta t}}]^b\,[e^{-(1+\rho)\sigma\sqrt{\Delta t}}]^f$
10. $S_0 \cdot [e^{(1-\rho)\sigma\sqrt{\Delta t}}]^c[e^{-(1-\rho)\sigma\sqrt{\Delta t}}]^d$	11. $S_0 \cdot [e^{(1-\rho)\sigma\sqrt{\Delta t}}]^c[e^{-\sigma\sqrt{\Delta t}}]^e$	12. $S_0 \cdot [e^{(1-\rho)\sigma\sqrt{\Delta t}}]^c\,[e^{-(1+\rho)\sigma\sqrt{\Delta t}}]^f$
13. $S_0 \cdot [e^{-(1-\rho)\sigma\sqrt{\Delta t}}]^d[e^{-\sigma\sqrt{\Delta t}}]^e$	14. $S_0 \cdot [e^{-(1-\rho)\sigma\sqrt{\Delta t}}]^d[e^{-(1+\rho)\sigma\sqrt{\Delta t}}]^f$	15. $S_0 \cdot [e^{-\sigma\sqrt{\Delta t}}]^e\,[e^{-(1+\rho)\sigma\sqrt{\Delta t}}]^f$

where $a + b + c + d + e + f = t - 1$, t is period, and $a, b, c, d, e, f = 1, 2, 3, \ldots$

We use several examples to describe the computation value of the fuzzy tree of stock price.

Example. see Appendix B.

When $t = 5, 6, \ldots, n$, we can deduce in the same way, too. Through the above explanation, we organize a general solution for the fuzzy binomial OPM to determine the number and value at every node at different periods. After we re-organize the same values, we can obtain the numbers of nodes and the corresponding values that are introduced as below.

If $t = 1$, we will get the current price is S_0, which $t - 1 = 0$, $N_1 = N_{1-1} + 4(1 - 1) + 1 = N_0 + 4(0) + 1 = 1$

If $t = 2$, we get $S_{ur}, S_{um}, S_{ul}, S_{dr}, S_{dm}$ and S_{dl}, which $t - 1 = 1$, $N_2 = N_{2-1} + 4(2-1) + 1 = N_1 + 4(1) + 1 = 1 + 4 + 1 = 6$

If $t = 3$, which $t - 1 = 2$, $N_3 = N_{3-1} + 4(3 - 1) + 1 = N_2 + 4(2) + 1 = 6 + 8 + 1 = 15$

If $t = 4$, which $t - 1 = 3$, $N_4 = N_{4-1} + 4(4 - 1) + 1 = N_3 + 4(3) + 1 = 15 + 12 + 1 = 28$

Next, we can also get the call price at $t = 5, 6, \ldots, n$ in the same way.

So we conclude the numbers of node at different periods as follows:

$$N_t = N_{t-1} + 4(t - 1) + 1$$

where,

N_t: the numbers of nodes at t

N_{t-1}: the numbers of nodes at $t - 1$

n: the number of periods, where $t = 1 \sim n$, and $N_0 = 0$

By following this methodology, we can get $N_5 = 45$, $N_6 = 66$, and we will get the numbers of nodes at different periods by following this methodology. As discussed earlier, we know that the movement of stock in the fuzzy binomial OPM is the same as with the CRR model, it either moves up or down. But there are three scenarios for each "up" and "down" in fuzzy binomial OPM. These values can also be computed accordingly.

4. Empirical analysis: Case of S&P 500 index options

We use actual cases to explain the process of fuzzy binomial OPM in this section. Firstly, we need to identify the variables and data characteristics, which are described as below.

4.1. Variables and data description

The data were obtained from the DataStream figures for S&P 500 index option prices, and its underlying security is the S&P 500 stock index. This sample data consists of daily closing S&P 500 index call options from July 28, 2003, to March 15, 2004. The stock price index and call price at issuing date are 996.52 and 16.4, respectively. The strike price of S&P 500 index is 1100. According Hull (1998), we should use the trading days instead of calendar days, the unit must be annualized, but the life of a call warrant is usually less than one year, so we use the 3 Month US Treasury Bill Rate to substitute for the riskless interest rate. The lowest interest rate (R_l) is 0.85%, the middle interest rate (R_m) is 0.915% and the highest interest rate (R_h) is 0.98%. Then we use historical data to estimate volatility of closing price from daily data over the most recent 30 to 60 days.

4.2. The fuzzy tree of stock price

We use historical data of CBOT (Chicago Board of Trade) to estimate volatility and use the riskless interest rate and R_l, R_m, R_h to estimate the lowest, middle and highest discount factor. Where $\sigma(1+\rho)$ and $\sigma(1-\rho)$ are the volatility of the right and left value of up movement, so $ur = [e^{(1+\rho)\sigma\sqrt{\Delta t}}]$, $um = [e^{\sigma\sqrt{\Delta t}}]$, and $ul = [e^{(1-\rho)\sigma\sqrt{\Delta t}}]$ are the degree of up movement, $S_{ur}(=ur\cdot S)$, $S_{um}(=um\cdot S)$, and $S_{ul}(=ul\cdot S)$ are the stock price of up movement. In other words, $-\sigma(1-\rho)$ and $-\sigma(1+\rho)$ are the volatility of the right and left value of down movement, the degree and the stock price of down movement can be computed in the same way.

For example, when the fuzzy interval $\rho = 5\%$, the number of division $t = 2$, σ is initial volatility, 1.05σ and 0.95σ are the volatility of right value and left value of up movement, -0.95σ and -1.05σ are the volatility of right value and left value of down movement. Then, we let "u" and "d" be the degrees of up and down movements, which have their own triangular fuzzy numbers.

The estimated volatility from historical data (2004/1/1–2004/3/15) is 11.4181%, after the fuzzy volatility, we get the stock price S_{ur}, S_{um} and S_{ul} under up movement and S_{dr}, S_{dm} and S_{dl} under down movement respectively. For instance, S_{ur} is the right value of up movement, where $ur = e^{(1+\rho)\sigma\sqrt{\Delta t}}$,so $S_{ur} = 996.52 \times e^{1.05\times\sigma\sqrt{0.3}} = 1064.1542$; S_{ul} is the left value of up movement, where $ul = e^{(1-\rho)\sigma\sqrt{\Delta t}}$,so $S_{ul} = 996.52 \times e^{0.95\times\sigma\sqrt{0.3}} =$

1057.5198. The stock price interval will be between 1064.1542 and 1057.5198 under up movement; if S_{dr} is the right value of down movement, where $dr = e^{-(1-\rho)\sigma\sqrt{\Delta t}}$, so $S_{dr} = 996.52 \times e^{-0.95\times\sigma\sqrt{0.3}} = 939.0388$; if S_{dl} is the left value of down movement, where $dl = e^{-(1+\rho)\sigma\sqrt{\Delta t}}$,so $S_{dl} = 996.52 \times e^{-1.05\times\sigma\sqrt{0.3}} = 933.1844$. Then we can get the result that the stock price interval will be between 939.0388 and 933.1844 under down movement. The concept of the fuzzy binomial OPM is much easier to comprehend, and it is much closer to reality than the CRR model. There are three scenarios for up and down movements, and six nodes on $t = 2$. We use the same method to fuzzily every node, and combining the same value, we can get 15 nodes on $t = 3$. The result is described as follows.

4.3. *The fuzzy interval of option price*

We use the stock price at $t = 0.6$ and exercise price to compute the option price at $t = 0.6$. By $C_{ur} = \max(S_{ur} - X, 0)$, $C_{um} = \max(S_{um} - X, 0)$, $C_{ul} = \max(S_{ul} - X, 0)$, $C_{dr} = \max(S_{dr} - X, 0)$, $C_{dm} = \max(S_{dm} - X, 0)$, $C_{dl} = \max(S_{dl} - X, 0)$. Next, after fuzzy the risk interest rate, we can get the optimal interval for option price of every node:$C_u = e^{-R_l\Delta t}[P_u \cdot C_{ur} + (1 - P_u) \cdot C_{dl}]$, $C_m = e^{-R_m\Delta t}[P_m \cdot C_{um} + (1 - P_m) \cdot C_{dm}]$ and $C_d = e^{-R_h\Delta t}[P_d \cdot C_{ul} + (1 - P_d) \cdot C_{dr}]$. For example, referring to Figure 2, C_{um} is max(1132.8306 − 1100,0) = 32.8306,C_{dm} is max(999.6410–1100,0) = 0, and through $C_m = e^{-r_m\Delta t}[P_m \cdot C_{um} + (1 - P_m) \cdot C_{dm}]$, we can get C_m is 16.5779. Next, we compute the *BNP* value of a triangular fuzzy number. The *BNP* value of C_{ur} is 18.3606, C_{dl} is 0 at $t = 0.3$, and through them we get the right value of current fuzzy option price, $C_u = 8.3672$ at $t = 0$. Using this method, we can get the current call price of triangular fuzzy numbers. To conduct the call price of sensitivity analysis, let there be four periods, every period is three to four month ($t = 0.3$). And the fuzzy interval sensitivity is 5%, 10%, 15%, and 20%. The variety of call prices can be divided as Table 1.

4.4. *Discussions*

Financial applications of fuzzy set have consumption behavior analysis (e.g. consumer preference matching; consumption decomposition; spending trend prediction; subjective price evaluation; prediction of satisfaction level), credit card business (e.g. credit fraud detection; automated fraud explanation; credit spending analysis), market forecast (e.g. stock price forecast; consumer price forecast; industry/regional growth prediction), market demand analysis, market fluctuation forecast, market simulations (e.g. pricing strategy study) and case studies (e.g. scenario simulations)... etc. So this study use fuzzy set in binomial OPM.

Because of financial investments are not motivated strictly by returns. To date, “high returns” and “risk aversion” remain to be two equally important objectives in investments. In order to establish a complete, valid “stock option” pricing model that provides a combined explanation for the both investment motivations, we need to adopt a two-dimensional “return-risk” space for the construction of the research and analysis environment. Whereas most investors face with the trade-offs and ambiguities between risk and return in making an investment decision, therefore a risk lover more concerns the return than the risk; he

Table 1. The variety of Call prices and sensitivity analysis

Sensitivity analysis (ρ)	Call prices of periods divided				
	1 ($t = 0.3$)	2 ($t = 0.6$)	Expiration dates ($t = 0.664$)	3 ($t = 0.9$)	4 ($t = 1.2$)
5%	8.3672	13.6326	14.9518	19.8162	25.4562
	7.4695	13.1544	14.4258	19.1139	24.9621
	6.5731	12.6761	13.8999	18.4125	24.4680
10%	9.2375	14.0695	15.4336	20.4637	25.8915
	7.4684	13.1553	14.4260	19.1115	24.9640
	5.6975	12.2425	13.4191	17.7580	24.0378
15%	10.1050	14.5087	15.9258	21.1513	26.3689
	7.4658	13.1573	14.4305	19.1253	24.9732
	4.8169	11.8104	12.9668	17.2311	23.6359
20%	10.9702	14.9881	16.4625	21.8991	27.0043
	7.4615	13.1605	14.4523	19.2159	25.0511
	3.9302	11.3984	12.5701	16.8905	23.3409

prefers the call price higher and chooses the right value of triangular fuzzy number. In this way, he will make more profit, because of the call price more fluctuate in high price. On the other hand, a risk averter more concerns the risk than the return. Because he considers hedge and loss, he chooses the left value of triangular fuzzy number, let his cost be lower that the loss will be less. A risk neutral considers both “profitability” and “stability”, so he chooses reasonable prices, which lie between the right value and left value of triangular fuzzy number. We know that the investor of the different risk preference will choose the different the triangular fuzzy number of market price. Therefore, the fuzzy triangular number of market price provides more choices for the investors and it can explain the risk preference of investor.

5. A comparison

To show the differences between generalized binomial OPM with fuzzy binomial OPM. The call prices of periods divided in generalized binomial OPM as Table 2.

Table 2. The variety of Call prices and sensitivity analysis

Sensitivity analysis (ρ)	Call prices of periods divided				
	1 ($t = 0.3$)	2 ($t = 0.6$)	Expiration dates ($t = 0.664$)	3 ($t = 0.9$)	4 ($t = 1.2$)
5%	9.0946	14.1880	15.7130	21.3362	26.6044
10%	10.8769	15.6178	17.4354	24.1379	29.0729
15%	12.6704	17.0609	19.1735	26.9636	31.5690
20%	14.4751	18.5176	20.9274	29.8135	34.0930

From Table 1, we know that the medium values of call prices are very stable to follow the variety of sensitivity at different periods, but when the sensitivity becomes larger, the bigger the fuzzy interval. From Table 2, we know that the smaller the sensitivity, the precision is getting better. In the other words, when sensitivity becomes larger the bigger the call prices. In this case, when sensitivity becomes larger, the fuzzy binominal model is more precise than the generalized CRR model, and it is much closer to the reality than the generalized CRR model. Because there are more nodes in the fuzzy binominal model, it becomes much closer to the continuous B-S model, and the convergence is faster than the generalized CRR model.

After applying fuzzy set theory to the generalized CRR model, we get (Table 1) for $t = 0.664$ the time until option expiration of triangular fuzzy number is between 13.9 and 14.95 in $\rho = 5\%$, and in the generalized CRR model is 15.713. Both values from CRR model and fuzzy binominal model are lower than the real market price of 16.4. This is due to the system bias between theoretical value and market price, so in practice, the market price is usually higher than theoretical value of an option (Black, 1975 and Merton, 1976). When sensitivity is bigger, for $t = 0.664$ the time until option expiration of triangular fuzzy number is between 12.57 and 16.46 in $\rho = 20\%$, but in the generalized CRR model is 20.9274. The interval values from fuzzy binominal model include the real market price of 16.4, but the generalized CRR model is higher than it. The fuzzy number is closer to real market price than CRR model. Compared with a generalized CRR model, the value of the generalized CRR model falls above the right value of triangular fuzzy number, so it is higher than the value of the right value of the triangular fuzzy number.

The impact of implicit “Fuzziness” is inevitable due to the subjective assessment made by investors in a generalized CRR model. How to draw the high or low of each criterions are described by linguistic terms, which can be expressed in triangular fuzzy numbers. The concept of triangular fuzzy number attempts to deal with real problems by possibility, it is much easier to comprehend. Owing to vague concepts frequently represented in decision data, the crisp values are inadequate to model real-life situations, so we can’t predict the call price exactly. In fact, it exist price interval. In fuzzy binomial OPM, the call price is an interval, so it is more elastic and also corresponds to actual situation. Investors can correct their portfolio strategy according to the right and left value of triangular fuzzy number, and they can interpret the optimal difference, according to their individual risk preferences.

6. Conclusions

When we wish to confirm and describe the price of an option, we can not exactly calculate its price, since the price of option is an interval under uncertainty environment. Investors who use the generalized CRR model, they must be able to estimate and forecast both riskless interest rate and volatility as the basis of analysis. It is inevitable that an investor’s estimation and forecasting by different degrees of uncertainty, therefore this paper attempts to construct a fuzzy binomial OPM to forecast the price of option. We propose a solution for the uncertainty investment question and promote the practicable of application in OPM.

According to the right and left value of triangular fuzzy number, investors are able to correct their portfolio strategy by combining CRR model and fuzzy set theory. When the market price lies between the right value and left value of triangular fuzzy number, a risk lover can buy much more, whereas a risk averter can buy less than those who are risk neutral. When market price is lower than the left value of triangular fuzzy number, a risk averter can buy more. Therefore, this research result provides more choices for the investors.

Appendix A: Fuzzy set theory

Fuzzy set theory was first introduced by Zadeh (1965), and was subsequently applied to various areas where two-valued logic was not reasonable. In reality, many things cannot be distinguished as vague but fuzzy way.

A.1. Triangular fuzzy number, TFN

The concept of triangular fuzzy number attempts to deal with real problems by possibility. The membership of a triangular fuzzy number is defined as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} (x-l)/(m-l) & m \le x \le r \\ (r-x)/(r-m) & l \le x \le m \\ 0 & \text{otherwise} \end{cases}$$

A.2. Fuzzy number

According to the characteristics of triangular fuzzy numbers and the extension principle put forward by Zadeh (1965), the operational laws of triangular fuzzy numbers, $\tilde{A} = (l_1, m_1, r_1)$ and $\tilde{B} = (l_2, m_2, r_2)$ are as follows:

(1) Addition of two fuzzy numbers$\oplus$

$$(l_1, m_1, r_1) \oplus (l_2, m_2, r_2) = (l_1 + l_2, m_1 + m_2, r_1 + r_2)$$

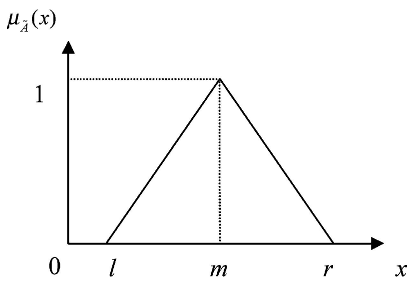

Figure 2. The membership function of the triangular fuzzy number.

(2) Subtraction of two fuzzy numbersΘ

$$(l_1, m_1, r_1) \Theta (l_2, m_2, r_2) = (l_1 - r_2, m_1 - m_2, r_1 - l_2)$$

(3) Multiplication of two fuzzy numbers $\otimes$

$$(l_1, m_1, r_1) \otimes (l_2, m_2, r_2) \cong (l_1 l_2, m_1 m_2, r_1 r_2)$$

(4) Multiplication of any real number k and a fuzzy number $\otimes$

$$k \otimes (l_1, m_1, r_1) = (kl_1, km_1, kr_1)$$

(5) Division of two fuzzy numbers $\oslash$

$$(l_1, m_1, r_1) \oslash (l_2, m_2, r_2) \cong (l_1/r_2, m_1/m_2, r_1/l_2)$$

A.3. Fuzzy relation

We use 0/1 to represent having an is/isn't relationship between elements in a traditional relation; and in contrast, we use a number between 0 and 1 to represent the relationship of elements in fuzzy relation. When the relationship is closer to 1, the relationship is stronger, and when the relationship is closer to 0, the relationship is weaker. In fuzzy set theory, we describe the relationship by a membership function.

These basis operators include joint, intersection and composition. Let ting $A \subseteq X \times Y$ and $B \subseteq X \times Y$, then the relative operations between A and B are as follows:

(1) Joint

$$\begin{aligned} A \cup B &= \mu_{A \cup B}(x, y) \\ &= \max[\mu_A(x, y), \mu_B(x, y)] \\ &= \vee[\mu_A(x, y), \mu_B(x, y)] \end{aligned}$$

(2) Intersection

$$\begin{aligned} A \cap B &= \mu_{A \cap B}(x, y) \\ &= \min[\mu_A(x, y), \mu_B(x, y)] \\ &= \wedge[\mu_A(x, y), \mu_B(x, y)] \end{aligned}$$

(3) Composition: There are several types of composition, of which the max-min operation is by far the most well-known. If $A \subseteq X \times Y$ and $B \subseteq X \times Y$, then the max-min operation of A and B is

$$\begin{aligned} A \circ B &= \mu_{A \circ B}(x, z) \\ &= \max[\min(\mu_A(x, y), \mu_B(y, z))] \\ &= \vee[\mu_A(x, y) \wedge \mu_B(y, z)] \end{aligned}$$

where "$\circ$" is an operator.

Appendix B

Example 1. If $t = 1, a + b + c + d + e + f = 0$, then $a = b = c = d = e = f = 0$, and the volatility of $[e^{(1+\rho)\sigma\sqrt{\Delta t}}]^0 = [e^{\sigma\sqrt{\Delta t}}]^0 = [e^{(1-\rho)\sigma\sqrt{\Delta t}}]^0 = [e^{-(1-\rho)\sigma\sqrt{\Delta t}}]^0 = [e^{-\sigma\sqrt{\Delta t}}]^0 = [e^{-(1+\rho)\sigma\sqrt{\Delta t}}]^0 = 1$. Therefore, we get the only value S_0.

Example 2. If $t = 2, a + b + c + d + e + f = 1$, there are the following 6 stock prices.

(1) $a = 1, b + c + d + e + f = 0$ then, $S_0 \cdot [e^{(1+\rho)\sigma\sqrt{\Delta t}}]^1$, the other items are zero;
(2) $b = 1, a + c + d + e + f = 0$ then, $S_0 \cdot [e^{\sigma\sqrt{\Delta t}}]^1$, the other items are zero;
(3) $c = 1, a + b + d + e + f = 0$ then, $S_0 \cdot [e^{(1-\rho)\sigma\sqrt{\Delta t}}]^1$, the other items are zero;
(4) $d = 1, a + b + c + e + f = 0$ then, $S_0 \cdot [e^{-(1-\rho)\sigma\sqrt{\Delta t}}]^1$, the other items are zero;
(5) $e = 1, a + b + c + d + f = 0$ then, $S_0 \cdot [e^{-\sigma\sqrt{\Delta t}}]^1$, the other items are zero;
(6) $f = 1, a + b + c + d + e = 0$ then, $S_0 \cdot [e^{-(1+\rho)\sigma\sqrt{\Delta t}}]^1$, the other items are zero.

Example 3. If $t = 3, a + b + c + d + e + f = 2$, there are the following 21 stock prices.

(1) $a = 2, b + c + d + e + f = 0$, then, $S_0 \cdot [e^{(1+\rho)\sigma\sqrt{\Delta t}}]^2 = S_0 \cdot e^{2(1+\rho)\sigma\sqrt{\Delta t}}$;
(2) $b = 2, a + c + d + e + f = 0$, then, $S_0 \cdot [e^{\sigma\sqrt{\Delta t}}]^2 = S_0 \cdot e^{2\sigma\sqrt{\Delta t}}$;
(3) $c = 2, a + b + d + e + f = 0$, then, $S_0 \cdot [e^{(1-\rho)\sigma\sqrt{\Delta t}}]^2 = S_0 \cdot e^{2(1-\rho)\sigma\sqrt{\Delta t}}$;
(4) $d = 2, a + b + c + e + f = 0$, then, $S_0 \cdot [e^{-(1-\rho)\sigma\sqrt{\Delta t}}]^2 = S_0 \cdot e^{-2(1-\rho)\sigma\sqrt{\Delta t}}$;
(5) $e = 2, a + b + c + d + f = 0$, then, $S_0 \cdot [e^{-\sigma\sqrt{\Delta t}}]^2 = S_0 \cdot e^{-2\sigma\sqrt{\Delta t}}$;
(6) $f = 2, a + b + c + d + e = 0$, then, $S_0 \cdot [e^{-(1+\rho)\sigma\sqrt{\Delta t}}]^2 = S_0 \cdot e^{-2(1+\rho)\sigma\sqrt{\Delta t}}$;
(7) $a = b = 1, c + d + e + f = 0$, then, $S_0 \cdot [e^{(1+\rho)\sigma\sqrt{\Delta t}}]^1[e^{\sigma\sqrt{\Delta t}}]^1 = S_0 \cdot e^{(2+\rho)\sigma\sqrt{\Delta t}}$;
(8) $a = c = 1, b + d + e + f = 0$, then, $S_0 \cdot [e^{(1+\rho)\sigma\sqrt{\Delta t}}]^1[e^{(1-\rho)\sigma\sqrt{\Delta t}}]^1 = S_0 \cdot e^{2\sigma\sqrt{\Delta t}}$;
(9) $a = d = 1, b + c + e + f = 0$, then, $S_0 \cdot [e^{(1+\rho)\sigma\sqrt{\Delta t}}]^1[e^{-(1-\rho)\sigma\sqrt{\Delta t}}]^1 = S_0 \cdot e^{2\rho\sigma\sqrt{\Delta t}}$;
(10) $a = e = 1, b + c + d + f = 0$, then, $S_0 \cdot [e^{(1+\rho)\sigma\sqrt{\Delta t}}]^1[e^{-\sigma\sqrt{\Delta t}}]^1 = S_0^{\rho\sigma\sqrt{\Delta t}}$;
(11) $a = f = 1, b + c + d + e = 0$, then, $S_0 \cdot [e^{(1+\rho)\sigma\sqrt{\Delta t}}]^1[e^{-(1+\rho)\sigma\sqrt{\Delta t}}]^1 = S_0$;
(12) $b = c = 1, a + d + e + f = 0$, then, $S_0 \cdot [e^{\sigma\sqrt{\Delta t}}]^1[e^{(1-\rho)\sigma\sqrt{\Delta t}}]^1 = S_0 \cdot e^{(2-\rho)\sigma\sqrt{\Delta t}}$;

(13) $b = d = 1, a + c + e + f = 0$, then, $S_0 \cdot [e^{\sigma\sqrt{\Delta t}}]^1[e^{-(1-\rho)\sigma\sqrt{\Delta t}}]^1 = S_0^{\rho\sigma\sqrt{\Delta t}}$;
(14) $b = e = 1, a + c + d + f = 0$, then, $S_0 \cdot [e^{\sigma\sqrt{\Delta t}}]^1[e^{-\sigma\sqrt{\Delta t}}]^1 = S_0$;
(15) $b = f = 1, a + c + d + e = 0$, then, $S_0 \cdot [e^{\sigma\sqrt{\Delta t}}]^1[e^{-(1+\rho)\sigma\sqrt{\Delta t}}]^1 = S_0 \cdot e^{-\rho\sigma\sqrt{\Delta t}}$;
(16) $c = d = 1, a + b + e + f = 0$, then, $S_0 \cdot [e^{(1-\rho)\sigma\sqrt{\Delta t}}]^1[e^{-(1-\rho)\sigma\sqrt{\Delta t}}]^1 = S_0$;
(17) $c = e = 1, a + b + d + f = 0$, then, $S_0 \cdot [e^{(1-\rho)\sigma\sqrt{\Delta t}}]^1[e^{-\sigma\sqrt{\Delta t}}]^1 = S_0 \cdot e^{-\rho\sigma\sqrt{\Delta t}}$;
(18) $c = f = 1, a+b+d+e = 0$, then, $S_0 \cdot [e^{(1-\rho)\sigma\sqrt{\Delta t}}]^1[e^{-(1+\rho)\sigma\sqrt{\Delta t}}]^1 = S_0 \cdot e^{-2\rho\sigma\sqrt{\Delta t}}$;
(19) $d = e = 1, a+b+c+f = 0$, then, $S_0 \cdot [e^{-(1-\rho)\sigma\sqrt{\Delta t}}]^1[e^{-\sigma\sqrt{\Delta t}}]^1 = S_0 \cdot e^{-(2-\rho)\sigma\sqrt{\Delta t}}$;
(20) $d = f = 1, a+b+c+e = 0$, then, $S_0 \cdot [e^{-(1-\rho)\sigma\sqrt{\Delta t}}]^1[e^{-(1+\rho)\sigma\sqrt{\Delta t}}]^1 = S_0 \cdot e^{-2\sigma\sqrt{\Delta t}}$;
(21) $e = f = 1, a+b+c+d = 0$, then, $S_0 \cdot [e^{-\sigma\sqrt{\Delta t}}]^1[e^{-(1+\rho)\sigma\sqrt{\Delta t}}]^1 = S_0 \cdot e^{-(2+\rho)\sigma\sqrt{\Delta t}}$.

After re-organizing the identical situations above, we get 15 unique situations:

1.	2.	3.	4.	5.	6.	7.	8.	9.	10	11.	12.	13.	14.	15.
1	7	2,8	12	3	9	10,13	11,14,16	15,17	18	4	19	20,5	21	6

Example 4. If $t = 4$, $a + b + c + d + e + f = 3$, we have 56 stock prices as follow:

(1) $a = 3, b + c + d + e + f = 0$, then, $S_0 \cdot [e^{(1+\rho)\sigma\sqrt{\Delta t}}]^3 = S_0 \cdot e^{3(1+\rho)\sigma\sqrt{\Delta t}}$;
(2) $b = 3, a + c + d + e + f = 0$, then, $S_0 \cdot [e^{\sigma\sqrt{\Delta t}}]^3 = S_0 \cdot e^{3\sigma\sqrt{\Delta t}}$;
(3) $c = 3, a + b + d + e + f = 0$, then, $S_0 \cdot [e^{(1-\rho)\sigma\sqrt{\Delta t}}]^3 = S_0 \cdot e^{3(1-\rho)\sigma\sqrt{\Delta t}}$;
(4) $d = 3, a + b + c + e + f = 0$, then, $S_0 \cdot [e^{-(1-\rho)\sigma\sqrt{\Delta t}}]^3 = S_0 \cdot e^{-3(1-\rho)\sigma\sqrt{\Delta t}}$;
(5) $e = 3, a + b + c + d + f = 0$, then, $S_0 \cdot [e^{-\sigma\sqrt{\Delta t}}]^3 = S_0 \cdot e^{-3\sigma\sqrt{\Delta t}}$;
(6) $f = 3, a + b + c + d + e = 0$, then, $S_0 \cdot [e^{-(1+\rho)\sigma\sqrt{\Delta t}}]^3 = S_0 \cdot e^{-3(1+\rho)\sigma\sqrt{\Delta t}}$;
(7) $a = 2, b = 1, c + d + e + f = 0$, then, $S_0 \cdot [e^{(1+\rho)\sigma\sqrt{\Delta t}}]^2[e^{\sigma\sqrt{\Delta t}}]^1 = S_0 \cdot e^{(3+2\rho)\sigma\sqrt{\Delta t}}$;
(8) $a = 2, c = 1, b+d+e+f = 0$, then, $S_0 \cdot [e^{(1+\rho)\sigma\sqrt{\Delta t}}]^2[e^{(1-\rho)\sigma\sqrt{\Delta t}}]^1 = S_0 \cdot e^{(3+\rho)\sigma\sqrt{\Delta t}}$;
(9) $a = 2, d = 1, b + c + e + f = 0$, then, $S_0 \cdot [e^{(1+\rho)\sigma\sqrt{\Delta t}}]^2[e^{-(1-\rho)\sigma\sqrt{\Delta t}}]^1 = S_0 \cdot e^{(1+3\rho)\sigma\sqrt{\Delta t}}$;
(10) $a = 2, e = 1, b + c + d + f = 0$, then, $S_0 \cdot [e^{(1+\rho)\sigma\sqrt{\Delta t}}]^2[e^{-\sigma\sqrt{\Delta t}}]^1 = S_0^{(1+2\rho)\sigma\sqrt{\Delta t}}$;
(11) $a = 2, f = 1, b + c + d + e = 0$, then, $S_0 \cdot [e^{(1+\rho)\sigma\sqrt{\Delta t}}]^2[e^{-(1+\rho)\sigma\sqrt{\Delta t}}]^1 = S_0 \cdot e^{(1+\rho)\sigma\sqrt{\Delta t}}$;
(12) $b = 2, c = 1, a + d + e + f = 0$, then, $S_0 \cdot [e^{\sigma\sqrt{\Delta t}}]^2[e^{(1-\rho)\sigma\sqrt{\Delta t}}]^1 = S_0 \cdot e^{(3-\rho)\sigma\sqrt{\Delta t}}$;
(13) $b = 2, d = 1, a+c+e+f = 0$, then, $S_0 \cdot [e^{\sigma\sqrt{\Delta t}}]^2[e^{-(1-\rho)\sigma\sqrt{\Delta t}}]^1 = S_0 \cdot e^{(1+\rho)\sigma\sqrt{\Delta t}}$;
(14) $b = 2, e = 1, a + c + d + f = 0$, then, $S_0 \cdot [e^{\sigma\sqrt{\Delta t}}]^2[e^{-\sigma\sqrt{\Delta t}}]^1 = S_0 \cdot e^{\sigma\sqrt{\Delta t}}$;
(15) $b = 2, f = 1, a+c+d+e = 0$, then, $S_0 \cdot [e^{\sigma\sqrt{\Delta t}}]^2[e^{-(1+\rho)\sigma\sqrt{\Delta t}}]^1 = S_0 \cdot e^{(1-\rho)\sigma\sqrt{\Delta t}}$;
(16) $c = 2, d = 1, a + b + e + f = 0$, then, $S_0 \cdot [e^{(1-\rho)\sigma\sqrt{\Delta t}}]^2[e^{-(1-\rho)\sigma\sqrt{\Delta t}}]^1 = S_0 \cdot e^{(1-\rho)\sigma\sqrt{\Delta t}}$;
(17) $c = 2, e = 1, a+b+d+f = 0$, then, $S_0 \cdot [e^{(1-\rho)\sigma\sqrt{\Delta t}}]^2[e^{-\sigma\sqrt{\Delta t}}]^1 = S_0 \cdot e^{(1-2\rho)\sigma\sqrt{\Delta t}}$;
(18) $c = 2, f = 1, a + b + d + e = 0$, then, $S_0 \cdot [e^{(1-\rho)\sigma\sqrt{\Delta t}}]^2[e^{-(1+\rho)\sigma\sqrt{\Delta t}}]^1 = S_0 \cdot e^{(1-3\rho)\sigma\sqrt{\Delta t}}$;

(19) $d = 2, e = 1, a+b+c+f = 0$, then, $S_0 \cdot [e^{-(1-\rho)\sigma\sqrt{\Delta t}}]^2[e^{-\sigma\sqrt{\Delta t}}]^1 = S_0 \cdot e^{-(2-\rho)\sigma\sqrt{\Delta t}}$;

(20) $d = 2, f = 1, a+b+c+e = 0$, then, $S_0 \cdot [e^{-(1-\rho)\sigma\sqrt{\Delta t}}]^2[e^{-(1+\rho)\sigma\sqrt{\Delta t}}]^1 = S_0 \cdot e^{-(3-\rho)\sigma\sqrt{\Delta t}}$;

(21) $e = 2, f = 1, a+b+c+d = 0$, then, $S_0 \cdot [e^{-\sigma\sqrt{\Delta t}}]^2[e^{-(1+\rho)\sigma\sqrt{\Delta t}}]^1 = S_0 \cdot e^{-(3+\rho)\sigma\sqrt{\Delta t}}$;

(22) $a = 1, b = 2, c+d+e+f = 0$, then, $S_0 \cdot [e^{(1+\rho)\sigma\sqrt{\Delta t}}]^1[e^{\sigma\sqrt{\Delta t}}]^2 = S_0 \cdot e^{(2+\rho)\sigma\sqrt{\Delta t}}$;

(23) $a = 1, c = 2, b+d+e+f = 0$, then, $S_0 \cdot [e^{(1+\rho)\sigma\sqrt{\Delta t}}]^1[e^{(1-\rho)\sigma\sqrt{\Delta t}}]^2 = S_0 \cdot e^{(3-\rho)\sigma\sqrt{\Delta t}}$;

(24) $a = 1, d = 2, b+c+e+f = 0$, then, $S_0 \cdot [e^{(1+\rho)\sigma\sqrt{\Delta t}}]^1[e^{-(1-\rho)\sigma\sqrt{\Delta t}}]^2 = S_0 \cdot e^{-(1-3\rho)\sigma\sqrt{\Delta t}}$;

(25) $a = 1, e = 2, b+c+d+f = 0$, then, $S_0 \cdot [e^{(1+\rho)\sigma\sqrt{\Delta t}}]^1[e^{-\sigma\sqrt{\Delta t}}]^2 = S_0^{-(1-\rho)\sigma\sqrt{\Delta t}}$;

(26) $a = 1, f = 2, b+c+d+e = 0$, then, $S_0 \cdot [e^{(1+\rho)\sigma\sqrt{\Delta t}}]^1[e^{-(1+\rho)\sigma\sqrt{\Delta t}}]^2 = S_0 \cdot e^{-(1+\rho)\sigma\sqrt{\Delta t}}$;

(27) $b = 1, c = 2, a+d+e+f = 0$, then, $S_0 \cdot [e^{\sigma\sqrt{\Delta t}}]^1[e^{(1-\rho)\sigma\sqrt{\Delta t}}]^2 = S_0 \cdot e^{(3-2\rho)\sigma\sqrt{\Delta t}}$;

(28) $b = 1, d = 2, a+c+e+f = 0$, then, $S_0 \cdot [e^{\sigma\sqrt{\Delta t}}]^1[e^{-(1-\rho)\sigma\sqrt{\Delta t}}]^2 = S_0^{-(1-2\rho)\sigma\sqrt{\Delta t}}$;

(29) $b = 1, e = 2, a+c+d+f = 0$, then, $S_0 \cdot [e^{\sigma\sqrt{\Delta t}}]^1[e^{-\sigma\sqrt{\Delta t}}]^2 = S_0 \cdot e^{-\sigma\sqrt{\Delta t}}$;

(30) $b = 1, f = 2, a+c+d+e = 0$, then, $S_0 \cdot [e^{\sigma\sqrt{\Delta t}}]^1[e^{-(1+\rho)\sigma\sqrt{\Delta t}}]^2 = S_0 \cdot e^{-(1+2\rho)\sigma\sqrt{\Delta t}}$;

(31) $c = 1, d = 2, a+b+e+f = 0$, then, $S_0 \cdot [e^{(1-\rho)\sigma\sqrt{\Delta t}}]^1[e^{-(1-\rho)\sigma\sqrt{\Delta t}}]^2 = S_0 \cdot e^{-(1-\rho)\sigma\sqrt{\Delta t}}$;

(32) $c = 1, e = 2, a+b+d+f = 0$, then, $S_0 \cdot [e^{(1-\rho)\sigma\sqrt{\Delta t}}]^1[e^{-\sigma\sqrt{\Delta t}}]^2 = S_0 \cdot e^{-(1+\rho)\sigma\sqrt{\Delta t}}$;

(33) $c = 1, f = 2, a+b+d+e = 0$, then, $S_0 \cdot [e^{(1-\rho)\sigma\sqrt{\Delta t}}]^1[e^{-(1+\rho)\sigma\sqrt{\Delta t}}]^2 = S_0 \cdot e^{-(1+3\rho)\sigma\sqrt{\Delta t}}$;

(34) $d = 1, e = 2, a+b+c+f = 0$, then, $S_0 \cdot [e^{-(1-\rho)\sigma\sqrt{\Delta t}}]^1[e^{-\sigma\sqrt{\Delta t}}]^2 = S_0 \cdot e^{-(3-\rho)\sigma\sqrt{\Delta t}}$;

(35) $d = 1, f = 2, a+b+c+e = 0$, then, $S_0 \cdot [e^{-(1-\rho)\sigma\sqrt{\Delta t}}]^1[e^{-(1+\rho)\sigma\sqrt{\Delta t}}]^2 = S_0 \cdot e^{-(3+\rho)\sigma\sqrt{\Delta t}}$;

(36) $e = 1, f = 2, a+b+c+d = 0$, then, $S_0 \cdot [e^{-\sigma\sqrt{\Delta t}}]^1[e^{-(1+\rho)\sigma\sqrt{\Delta t}}]^2 = S_0 \cdot e^{-(3+2\rho)\sigma\sqrt{\Delta t}}$;

(37) $a = 1, b = 1, c = 1, d+e+f = 0$, then, $S_0 \cdot [e^{(1+\rho)\sigma\sqrt{\Delta t}}]^1[e^{\sigma\sqrt{\Delta t}}]^1[e^{(1-\rho)\sigma\sqrt{\Delta t}}]^1 = S_0 \cdot e^{3\sigma\sqrt{\Delta t}}$;

(38) $a = 1, b = 1, d = 1, c+e+f = 0$, then, $S_0 \cdot [e^{(1+\rho)\sigma\sqrt{\Delta t}}]^1[e^{\sigma\sqrt{\Delta t}}]^1[e^{-(1-\rho)\sigma\sqrt{\Delta t}}]^1 = S_0^{(1+2\rho)\sigma\sqrt{\Delta t}}$;

(39) $a = 1, b = 1, e = 1, c+d+f = 0$, then, $S_0 \cdot [e^{(1+\rho)\sigma\sqrt{\Delta t}}]^1[e^{\sigma\sqrt{\Delta t}}]^1[e^{-\sigma\sqrt{\Delta t}}]^1 = S_0 \cdot e^{(1+\rho)\sigma\sqrt{\Delta t}}$;

(40) $a = 1, b = 1, f = 1, c+d+e = 0$, then, $S_0 \cdot [e^{(1+\rho)\sigma\sqrt{\Delta t}}]^1[e^{\sigma\sqrt{\Delta t}}]^1[e^{-(1+\rho)\sigma\sqrt{\Delta t}}]^1 = S_0 \cdot e^{\sigma\sqrt{\Delta t}}$;

(41) $a = 1, c = 1, d = 1, b+e+f = 0$, then, $S_0 \cdot [e^{(1+\rho)\sigma\sqrt{\Delta t}}]^1[e^{(1-\rho)\sigma\sqrt{\Delta t}}]^1[e^{-(1-\rho)\sigma\sqrt{\Delta t}}]^1 = S_0 \cdot e^{(1+\rho)\sigma\sqrt{\Delta t}}$;

(42) $a = 1, c = 1, e = 1, b+d+f = 0$, then, $S_0 \cdot [e^{(1+\rho)\sigma\sqrt{\Delta t}}]^1[e^{(1-\rho)\sigma\sqrt{\Delta t}}]^1[e^{-\sigma\sqrt{\Delta t}}]^1 = S_0 \cdot e^{\sigma\sqrt{\Delta t}}$;

(43) $a = 1, c = 1, f = 1, b+d+e = 0$, then, $S_0 \cdot [e^{(1+\rho)\sigma\sqrt{\Delta t}}]^1[e^{(1-\rho)\sigma\sqrt{\Delta t}}]^1[e^{-(1+\rho)\sigma\sqrt{\Delta t}}]^1 = S_0 \cdot e^{(1-\rho)\sigma\sqrt{\Delta t}}$;

(44) $a = 1, d = 1, e = 1, b+c+f = 0$, then, $S_0 \cdot [e^{(1+\rho)\sigma\sqrt{\Delta t}}]^1[e^{-(1-\rho)\sigma\sqrt{\Delta t}}]^1[e^{-\sigma\sqrt{\Delta t}}]^1 = S_0^{-(1-2\rho)\sigma\sqrt{\Delta t}}$;

(45) $a = 1, d = 1, f = 1, b+c+e = 0$, then, $S_0 \cdot [e^{(1+\rho)\sigma\sqrt{\Delta t}}]^1[e^{-(1-\rho)\sigma\sqrt{\Delta t}}]^1$ $[e^{-(1+\rho)\sigma\sqrt{\Delta t}}]^1 = S_0^{-(1-\rho)\sigma\sqrt{\Delta t}}$;

(46) $a = 1, e = 1, f = 1, b+c+d = 0$, then, $S_0 \cdot [e^{(1+\rho)\sigma\sqrt{\Delta t}}]^1[e^{-\sigma\sqrt{\Delta t}}]^1[e^{-(1+\rho)\sigma\sqrt{\Delta t}}]^1 = S_0 \cdot e^{-\sigma\sqrt{\Delta t}}$;

(47) $b = 1, c = 1, d = 1, a+e+f = 0$, then, $S_0 \cdot [e^{\sigma\sqrt{\Delta t}}]^1[e^{(1-\rho)\sigma\sqrt{\Delta t}}]^1[e^{-(1-\rho)\sigma\sqrt{\Delta t}}]^1 = S_0 \cdot e^{\sigma\sqrt{\Delta t}}$;

(48) $b = 1, c = 1, e = 1, a+d+f = 0$, then, $S_0 \cdot [e^{\sigma\sqrt{\Delta t}}]^1[e^{(1-\rho)\sigma\sqrt{\Delta t}}]^1[e^{-\sigma\sqrt{\Delta t}}]^1 = S_0 \cdot e^{(1-\rho)\sigma\sqrt{\Delta t}}$;

(49) $b = 1, c = 1, f = 1, a+d+e = 0$, then, $S_0 \cdot [e^{\sigma\sqrt{\Delta t}}]^1[e^{(1-\rho)\sigma\sqrt{\Delta t}}]^1[e^{-(1+\rho)\sigma\sqrt{\Delta t}}]^1 = S_0 \cdot e^{(1-2\rho)\sigma\sqrt{\Delta t}}$;

(50) $b = 1, d = 1, e = 1, a+c+f = 0$, then, $S_0 \cdot [e^{\sigma\sqrt{\Delta t}}]^1[e^{-(1-\rho)\sigma\sqrt{\Delta t}}]^1[e^{-\sigma\sqrt{\Delta t}}]^1 = S_0^{-(1-\rho)\sigma\sqrt{\Delta t}}$;

(51) $b = 1, d = 1, f = 1, a+c+e = 0$, then, $S_0 \cdot [e^{\sigma\sqrt{\Delta t}}]^1[e^{-(1-\rho)\sigma\sqrt{\Delta t}}]^1[e^{-(1+\rho)\sigma\sqrt{\Delta t}}]^1 = S_0 \cdot e^{-\sigma\sqrt{\Delta t}}$;

(52) $b = 1, e = 1, f = 1, a+c+d = 0$, then, $S_0 \cdot [e^{\sigma\sqrt{\Delta t}}]^1[e^{-\sigma\sqrt{\Delta t}}]^1[e^{-(1+\rho)\sigma\sqrt{\Delta t}}]^1 = S_0 \cdot e^{-(1+\rho)\sigma\sqrt{\Delta t}}$;

(53) $c = 1, d = 1, e = 1, a+b+f = 0$, then, $S_0 \cdot [e^{(1-\rho)\sigma\sqrt{\Delta t}}]^1[e^{-(1-\rho)\sigma\sqrt{\Delta t}}]^1[e^{-\sigma\sqrt{\Delta t}}]^1 = S_0 \cdot e^{-\sigma\sqrt{\Delta t}}$;

(54) $c = 1, d = 1, f = 1, a+b+e = 0$, then, $S_0 \cdot [e^{(1-\rho)\sigma\sqrt{\Delta t}}]^1[e^{-(1-\rho)\sigma\sqrt{\Delta t}}]^1$ $[e^{-(1+\rho)\sigma\sqrt{\Delta t}}]^1 = S_0 \cdot e^{-(1+\rho)\sigma\sqrt{\Delta t}}$;

(55) $c = 1, e = 1, f = 1, a+b+d = 0$, then, $S_0 \cdot [e^{(1-\rho)\sigma\sqrt{\Delta t}}]^1[e^{-\sigma\sqrt{\Delta t}}]^1[e^{-(1+\rho)\sigma\sqrt{\Delta t}}]^1 = S_0 \cdot e^{-(1+2\rho)\sigma\sqrt{\Delta t}}$;

(56) $d = 1, e = 1, f = 1, a+b+c = 0$, then, $S_0 \cdot [e^{-(1-\rho)\sigma\sqrt{\Delta t}}]^1[e^{-\sigma\sqrt{\Delta t}}]^1[e^{-(1+\rho)\sigma\sqrt{\Delta t}}]^1 = S_0 \cdot e^{-3\sigma\sqrt{\Delta t}}$.

After re-organizing the identical situations, we get 28 unique situations.

1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.	14.
1	7	8,22	37,2	23,12	27	3	9	10,38	11,39,41,13	40,42,14,47	43,15,48,16	49,17	18

15.	16.	17.	18.	19.	20.	21.	22.	23.	24.	25.	26.	27.	28.
24	44,28	45,25,50,27	46,51,29,53	26,52,54,32	30,55	33	4	19	20,34	56,5	35,21	36	6

Note

1. As mentioned earlier, except for the case of universal existence of such demand factors applicable to each and every investor, these factors have to undergo stock pricing standardization.

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